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A MODEL FOR A COMPOSITE WITH ANISOTROPIC DISSIPATION BY HOMOGENIZATION

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Abstract—A composite material, made up of linearly elastic inclusions and matrix, is considered. The two components are held together by viscous coupling. We show that the effective properties of this medium can be obtained using homogenization techniques. The resulting homogenized material is anisotropic in elasticities as well as in dissipation. For the special case of a laminated composite, we can state the effective constitutive law explicitly. The anisotropies in the dissipation are studied in detail.

1. INTRODUCTION

We model the propagation of waves in a two-dimensional medium which contains micro-structural inclusions. The latter are coupled to the matrix material by viscous contact. This simple-minded model is a first attempt at mathematically modeling propagation phenomena in a medium where there is inherent anisotropy in the dissipation. Possible applications include the description of wave propagation in damaged composites where substantial debonding of the fibers has occurred, and in geological formations consisting of rocks or plates which slide against one another.

The components of the medium in question will be materials satisfying linear isotropic elasticity. In addition, the microstructure will be assumed to be periodic. The underlying assumptions will be that the dominant wavelength of the disturbance is an order of magnitude larger than the lengthscale of the microstructure. We shall use homogenization techniques to show that it is possible to replace the complicated medium by an effective medium. The steps leading to homogenization will be presented.

A special case, that of a medium made up of fine layers, will be studied in detail. This further simplification allows us to write down the equations of the effective medium explicitly, and to give a description of its properties.

2. MODEL OF THE MEDIUM

We assume that our composite is made up of a mixture of two linearly elastic, isotropic components. To represent the inhomogeneity due to the microstructure, which is of length scale ε (small), it is convenient to define the Lamé moduli and density as functions of position $\mathbf{x} = (x_1, x_2)$. To incorporate the smallness of the microstructure, we denote the Lamé parameters and the density by

$$\mu^\varepsilon(\mathbf{x}) = \mu(\mathbf{x}/\varepsilon),$$
$$\lambda^\varepsilon(\mathbf{x}) = \lambda(\mathbf{x}/\varepsilon),$$
$$\rho^\varepsilon(\mathbf{x}) = \rho(\mathbf{x}/\varepsilon).$$

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Furthermore, the medium is periodic, with period $\mathbf{p}$:

$$\mu(\mathbf{y}+\mathbf{p}) = \mu(\mathbf{y}),$$
$$\lambda(\mathbf{y}+\mathbf{p}) = \lambda(\mathbf{y}),$$
$$\rho(\mathbf{y}+\mathbf{p}) = \rho(\mathbf{y}),$$

for any vector $\mathbf{y}$. Here the fixed vector $\mathbf{p} = (p_1, p_2)$ represents the periodic cell.

We shall work with particle displacements. Thus let $\mathbf{u}^\varepsilon$ be the particle displacement vector (components u_i^ε, $i = 1, 2$). We shall use $\boldsymbol{\sigma}^\varepsilon$ to represent the stress tensor (components σ_{ij}^ε). The superscript ε is used to indicate the dependence of the displacement and stress on the size of the microstructural cell. Hooke's law for isotropic elasticity states that

$$\sigma_{ij}^\varepsilon = \mu^\varepsilon(\partial_i u_j^\varepsilon + \partial_j u_i^\varepsilon) + \lambda^\varepsilon(\partial_k u_k^\varepsilon)\delta_{ij}. \tag{1}$$

We have used the standard notation: $\partial_i = \partial_{x_i}$.

We shall now go into a cell and describe the microstructure. For convenience, let this cell occupy $[0, \varepsilon p_1] \times [0, \varepsilon p_2]$. We shall use a scaled variable $\mathbf{y} = \mathbf{x}/\varepsilon$. We separate each cell into two regions, whose boundary is a closed curve Γ. The domain enclosed by Γ is denoted by Ω. The situation is depicted in Fig. 1. In each cell we assign the Lamé parameters to take on values:

$$\mu(\mathbf{y}) = \begin{cases} \mu^A & \text{if } \mathbf{y} \in \Omega \\ \mu^B & \text{if } \mathbf{y} \notin \Omega, \end{cases}$$

$$\lambda(\mathbf{y}) = \begin{cases} \lambda^A & \text{if } \mathbf{y} \in \Omega \\ \lambda^B & \text{if } \mathbf{y} \notin \Omega. \end{cases}$$

Likewise, we have for the density

$$\rho(\mathbf{y}) = \begin{cases} \rho^A & \text{if } \mathbf{y} \in \Omega \\ \rho^B & \text{if } \mathbf{y} \notin \Omega. \end{cases}$$

To complete the description of the microstructure, we shall precribe boundary conditions on Γ. To model the possibility of sliding along the boundary Γ which separates the two mixture components, we shall assume that the displacement normal to the boundary is continuous, while the velocity tangent to the boundary is allowed to suffer a jump whose

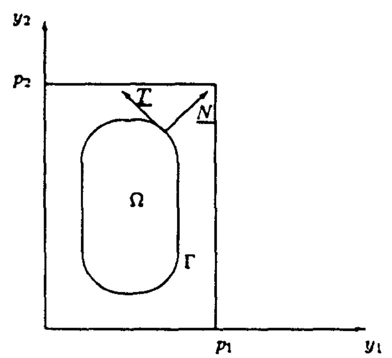

Fig. 1. This figure illustrates the geometry of a cell. The medium in question is made up periodically of identical cells.

magnitude depends linearly on the stress at that point. For the cell in question, these requirements are reflected in the equations

$$(u_i^\varepsilon|_{\Gamma+} - u_i^\varepsilon|_{\Gamma-})N_i = 0, \tag{2}$$

$$\frac{c}{\varepsilon}\partial_t(u_i^\varepsilon|_{\Gamma+} - u_i^\varepsilon|_{\Gamma-})T_i = \sigma_{ij}^\varepsilon N_i T_j|_{\Gamma\pm}. \tag{3}$$

Here, $\mathbf{N}$ and $\mathbf{T}$ are the outward normal and tangent to the curve Γ. The notation $\Gamma-$ implies that we approach the boundary from inside Ω, and $\Gamma+$ from outside Ω. The parameter c is the viscous constant.

The factor $1/\varepsilon$ on the left-hand-side above is necessary because we are interested in the behavior of the solution $\mathbf{u}^\varepsilon$ as $\varepsilon \to 0$. If we do not have this factor, the homogenized medium does not support propagation. This can be demonstrated using the asymptotic method employed in the next section.

The final equations needed to completely specify the problem are the balance of momentum equations, written here as

$$\rho^\varepsilon \partial_t^2 u_i^\varepsilon = \partial_j \sigma_{ij}^\varepsilon. \tag{4}$$

It is understood that the prescription of the cell problem is repeated until the entire domain of interest is covered. The problem we have in mind for this system will be some kind of initial–boundary value problem. Clearly the problem we have at hand is a very difficult one. It is not obvious for example how one should solve it numerically. However, in this work, we shall use a technique that allows us to investigate the behavior of the solution as $\varepsilon \to 0$.

3. HOMOGENIZATION

To analyze the problem at hand, we shall use a technique which is known as *homogenization methods* or *effective medium theory*. For excellent general references on this approach, see Bensoussan *et al.* (1978), and Sanchez-Palencia (1980); see also Sanchez-Palencia and Zaoui (1986) for a survey of examples of using homogenization to solve some engineering problems.

Homogenization attempts to find the effective behavior of the composite medium by looking at the limit when the size of the microstructure ε goes to 0. For hyperbolic problems that are in consideration here, this means that the dominant wavelength of the distrubance in the medium must be an order of magnitude larger than the length scale of the microstructure. This is a more 'standard' interpretation of homogenization, and is called the *static limit* by Bensoussan *et al.* (1978). For transient problems involving a wide spectrum in the disturbances, other techniques must be developed.

However, homogenization does give us a relatively simple way of studying this problem. In addition, the resulting effective medium equations are amenable to computation. It is an approximation whose accuracy has been assessed for the case of a porous medium, both in numerical simulations and in laboratory experiments—see Auriault *et al.* (1985). The issue of numerical analysis and accuracy of homogenization of elliptic partial differential equations has also been studied; see for example Babuska (1976) and Vogelius and Papanicolaou (1982).

It should be pointed out that homogenization is not the only technique available to obtain the effective properties of a composite. Achenbach and Sun (1971) presented a method based on averaging of physical quantities and requiring specific physical principles to be satisfied. While this procedure is intuitively appealing, homogenization has the advantage that it is a mathematical technique based on asymptotics. This fact allows one to proceed formally, and to apply it over a wide variety of physical problems (linear or nonlinear) described by partial differential equations.

Others have considered this problem of hyperbolic partial differential equations with period coefficients from another point of view. The fact that the coefficients are periodic allows one to use a version of Floquet Theory. The solution of the problem can usually be represented in terms of eigenfunctions (called Bloch waves). The theory, although very cumbersome computationally, is exact. For instance, it is possible to find the dispersion relation of the medium. Interested readers are referred to Achenbach and Kitahara (1987), Delph *et al.* (1979) and Odeh and Keller (1964). It seems possible to extend this theory to our problem, where in addition to the periodic coefficients, we also have internal boundary conditions.

It is also possible to break away from the assumption of medium periodicity by considering a random medium with microstructures. Burridge *et al.* (1987) studied long-time statistics of the response of a one-dimensional random medium. The problem of heat conduction in a medium with random conductivities was considered by Papanicolaou and Varadhan (1982), using a technique which is in the same spirit as homogenization. It was not clear however how the formulae could be exploited, as they appear to be very special and difficult to compute with.

The idea behind homogenization is to first assume that the solution depends on two spatial variables $\mathbf{x}$ and $\mathbf{y} = \mathbf{x}/\varepsilon$. The second variable, referred to as the 'fast variable', takes into account the two scales present in the problem. Let us suppress the dependence of the solution on t for the moment. The physical displacements and stresses are understood as

$$\mathbf{u}^{\varepsilon}(\mathbf{x}) = \mathbf{u}(\mathbf{x},\mathbf{y})|_{\mathbf{y}=\mathbf{x}/\varepsilon},$$
$$\boldsymbol{\sigma}^{\varepsilon}(\mathbf{x}) = \boldsymbol{\sigma}(\mathbf{x},\mathbf{y})|_{\mathbf{y}=\mathbf{x}/\varepsilon},$$

where both $\mathbf{u}$ and $\boldsymbol{\sigma}$ are functions of the two independent spatial variables $\mathbf{x}$ and $\mathbf{y}$.

Next, we write a power series expansion in ε:

$$\mathbf{u}(\mathbf{x},\mathbf{y}) = \mathbf{u}^0(\mathbf{x},\mathbf{y}) + \varepsilon\mathbf{u}^1(\mathbf{x},\mathbf{y}) + \varepsilon^2\mathbf{u}^2(\mathbf{x},\mathbf{y}) + \cdots, \tag{5}$$

$$\boldsymbol{\sigma}(\mathbf{x},\mathbf{y}) = \boldsymbol{\sigma}^0(\mathbf{x},\mathbf{y}) + \varepsilon\boldsymbol{\sigma}^1(\mathbf{x},\mathbf{y}) + \varepsilon^2\boldsymbol{\sigma}^2(\mathbf{x},\mathbf{y}) + \cdots. \tag{6}$$

It is further assumed that $\mathbf{u}^k$ and $\boldsymbol{\sigma}^k$ are periodic with period $\mathbf{p}$ in the variable $\mathbf{y}$ to take into account the periodicity of the microstructure.

Partial differentiation with respect to x_i of u^k and $\boldsymbol{\sigma}^k$ yields

$$\partial_i \rightarrow \partial_i + \frac{1}{\varepsilon} D_i, \tag{7}$$

where $D_i = \partial_{y_i}$.

The next step is simply to substitute the expansions in (5) and (6) into eqns (1) and (4), and match equal powers in ε. From eqns (1) we get

$$\mu(D_i u_j^0 + D_j u_i^0) + \lambda(D_k u_k^0)\delta_{ij} = 0, \tag{8}$$

from the ε^{-1} terms. From ε^0 terms, we get

$$\sigma_{ij}^0 = \mu(\partial_i u_j^0 + \partial_j u_i^0) + \lambda(\partial_k u_k^0)\delta_{ij} + \mu(D_i u_j^1 + D_j u_i^1) + \lambda(D_k u_k^1)\delta_{ij}. \tag{9}$$

Similarly, from equation (4), we obtain

$$D_j \sigma_{ij}^0 = 0, \tag{10}$$

$$\rho\partial_t^2 u_i^0 = \partial_j \sigma_{ij}^0 + D_j \sigma_{ij}^1. \tag{11}$$

From eqns (8), we can conclude that $\mathbf{u}^0$ depends only on $\mathbf{x}$ and t, because both λ and μ are both positive.

To make the following calculations more manageable, we will need to work with the strain tensor

$$\varepsilon_{ij}^0 = \tfrac{1}{2}(\partial_i u_j^0 + \partial_j u_i^0),$$

and the 'microstructural stress tensor'

$$\tau_{ij}^1 = \mu(D_i u_j^1 + D_j u_i^1) + \lambda(D_k u_k^1)\delta_{ij}.$$

Keep in mind that the Lamé moduli are functions of $\mathbf{y}$ only.

Next, we combine eqns (10) and (9) to obtain

$$D_j \tau_{ij}^1 = -[2D_j \mu \delta_{ik}\delta_{jl} + D_j \lambda \delta_{ij}\delta_{kl}]\varepsilon_{kl}^0. \tag{12}$$

To completely determine $\mathbf{u}^1$, we go to the boundary conditions (2) and (3). Using the expansion and matching equal powers of ε, we get

$$(u_i^1|_{\Gamma+} - u_i^1|_{\Gamma-})N_i = 0, \tag{13}$$

$$c\partial_t(u_i^1|_{\Gamma+} - u_i^1|_{\Gamma-})T_i = \sigma_{ij}^0 N_i T_j|_{\Gamma\pm}. \tag{14}$$

The last three equations, along with the periodicity requirement in the variable $\mathbf{y}$, determine $\mathbf{u}^1$.

To display the character of the effective medium, we need to obtain a representation for $\mathbf{u}^1$. To this end, we put eqn (9) in the right-hand-side of (14) and rearrange to get

$$c\partial_t(u_i^1|_{\Gamma+} - u_i^1|_{\Gamma-})T_i - \tau_{ij}^1 T_i N_j|_{\Gamma\pm} = (2\mu\delta_{ik}\delta_{jl} + \lambda\delta_{ij}\delta_{kl})T_i N_j|_{\Gamma\pm}\varepsilon_{kl}^0. \tag{15}$$

Notice that in the right-hand-side, ε_{kl}^0 depends only on $\mathbf{x}$ and t.

Now we are ready to define what are called the 'local problems' by Sanchez-Palencia (1980). Let the auxiliary functions $\boldsymbol{\chi}^{kl}(\mathbf{y})$ and $\boldsymbol{\eta}^{kl}(\mathbf{y}, t)$ be p-peridoc vector functions in $\mathbf{y}$ and satisfy

$$D_j(\mu(D_i\chi_j^{kl} + D_j\chi_i^{kl}) + \lambda(D_m\chi_m^{kl})\delta_{ij}) = -[2D_j\mu\delta_{ik}\delta_{jl} + D_j\lambda\delta_{ij}\delta_{kl}], \tag{16}$$

with $\boldsymbol{\chi}^{kl}$ continuous across Γ, and

$$D_j(\mu(D_i\eta_j^{kl} + D_j\eta_i^{kl}) + \lambda(D_m\eta_m^{kl})\delta_{ij}) = 0.$$

$$(\eta_i^{kl}|_{\Gamma+} - \eta_i^{kl}|_{\Gamma-})N_i = 0,$$

$$c\partial_t(\eta_i^{kl}|_{\Gamma+} - \eta_i^{kl}|_{\Gamma-})T_i - (\mu(D_i\eta_j^{kl} + D_j\eta_i^{kl}) + \lambda(D_m\eta_m^{kl})\delta_{ij}) = (2\mu\delta_{ik}\delta_{jl} + \lambda\delta_{ij}\delta_{kl})T_i N_j|_{\Gamma\pm}H(t). \tag{17}$$

$H(t)$ is the Heaviside function. The initial condition for $\boldsymbol{\eta}^{kl}$ should be one that agrees with the displacement vector. For instance, if $\mathbf{u}^\varepsilon = 0$ for $t < 0$, then we must have $\boldsymbol{\eta}^{kl} = 0$ for $t < 0$ also.

Then, from eqns (12), (14) and (15), we arrive at the representation

$$\mathbf{u}^1 = \boldsymbol{\chi}^{kl}(\mathbf{y})\varepsilon_{kl}^0 + \int_0^t \mathrm{d}s\,\boldsymbol{\eta}^{kl}(\mathbf{y}, t-s)\varepsilon_{kl}^0(s). \tag{18}$$

We can now write down the equations governing $\mathbf{u}^0$ and $\boldsymbol{\sigma}^0$. We shall use

$$\overline{(\cdot)} = \frac{1}{P}\int_P \mathrm{d}\mathbf{y}(\cdot)$$

to denote the averaging operation over a periodic cell P. From eqns (9) and (18), we have

$$\bar{\sigma}^0_{ij}(\mathbf{x},t) = \overline{[(2\mu\delta_{ik}\delta_{jl}+\lambda\delta_{ij}\delta_{kl})+\mu(D_i\chi^{kl}_j+D_j\chi^{kl}_i)+\lambda(D_m\chi^{kl}_m)\delta_{ij}]}\,\varepsilon^0_{kl}(\mathbf{x},t)$$
$$+\overline{[\mu(D_i\eta^{kl}_j+D_j\eta^{kl}_i)+\lambda(D_m\eta^{kl}_m)\delta_{ij}]} * \varepsilon^0_{kl}(\mathbf{x},t). \quad (19)$$

The equilibrium equation is obtained from (11):

$$\bar{\rho}\partial_t^2 u_i^0 = \partial\bar{\sigma}^0_{ij} \quad (20)$$

because $\overline{D_j\sigma^1_{ij}} = 0$ by periodicity.

Equations (19) and (20) specify the effective medium. It is thought of formally as the solution of the original problem for $\varepsilon \to 0$. The specification of the microstructure, that is, c, ρ, λ, μ and Γ, determines the auxiliary functions χ^{kl} and η^{kl}. Once they are obtained, they are inserted into eqns (19), which gives the effective coefficients of the lossy elastic medium.

The expression (19) has a very natural interpretation. The first term on the right is the elastic anisotropy contribution, due to the microstructural inhomogeneity of the medium. The second term is the contribution from the viscous losses caused by the presence of microstructural boundaries. Notice that the latter is a convolution of a kernel with the strain tensor (not a product!). It is possible to show that the effective Hooke's tensor satisfies the usual symmetries; see for example Sanchez-Palencia (1980). It is not very hard (only tedious) to show that the dissipation tensor η^{kl} is symmetric in k and l.

The second term is responsible for energy loss—this may be possible to assert from the nature of the kernel η^{kl} and energy estimates. However, at this point, we feel it will be more instructive to solve an explicit example—that of one-dimensional lamination, where energy decay does indeed take place.

4. ONE-DIMENSIONAL LAMINATION AND EFFECTIVE MEDIUM PROPERTIES

We specialize now to the case where the microstructure is a lamination. For convenience, we choose x_1 to be the layering direction. The periodic cell, here a slab, is of size ε. We need only a scalar 'fast' variable $y_1 = x_1/\varepsilon$. Let us concentrate our attention on the cell lying on the interval $[-\varepsilon/2, \varepsilon/2]$. In the y_1 coordinate, the interval $[a, b]$ is the domain Ω, Γ consists of two points $y_1 = a$ and $y_1 = b$. See Fig. 2.

The mixture in this cell is described by

$$\mu(y_1) = \begin{cases} \mu^A & \text{if } y_1 \in [a,b] \\ \mu^B & \text{if } y_1 \notin [a,b], \end{cases}$$

$$\lambda(y_1) = \begin{cases} \lambda^A & \text{if } y_1 \in [a,b] \\ \lambda^B & \text{if } y_1 \notin [a,b], \end{cases}$$

$$\rho(y_1) = \begin{cases} \rho^A & \text{if } y_1 \in [a,b] \\ \rho^B & \text{if } y_1 \notin [a,b]. \end{cases}$$

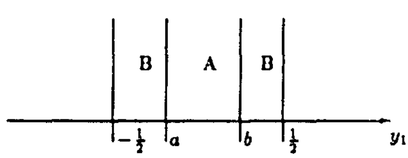

Fig. 2. The periodic cell for one-dimensional lamination is the interval $[-\frac{1}{2}, \frac{1}{2}]$. The interfaces are at $y_1 = a, b$. The slab occupying $[a, b]$ is mixture A, the remainder is mixture B.

The internal boundary conditions (2) and (3) reduce to

$$\frac{c}{\varepsilon}\partial_t[u_2^\varepsilon|_{a+} - u_2^\varepsilon|_{a-}] = \sigma_{12}^\varepsilon|_{a\pm}, \tag{21}$$

$$\frac{c}{\varepsilon}\partial_t[u_2^\varepsilon|_{b+} - u_2^\varepsilon|_{b-}] = \sigma_{12}^\varepsilon|_{b\pm}, \tag{22}$$

with continuity requirements on u_1^ε across $y_1 = a, b$.

Due to the one-dimensional lamination, the expansions (5) and (6) now take the form

$$\mathbf{u}^\varepsilon(\mathbf{x}) = \mathbf{u}^0(\mathbf{x}, y_1) + \varepsilon\mathbf{u}^1(\mathbf{x}, y_1) + \varepsilon^2\mathbf{u}^2(\mathbf{x}, y_1) + \cdots,$$

$$\boldsymbol{\sigma}^\varepsilon(\mathbf{x}) = \boldsymbol{\sigma}^0(\mathbf{x}, y_1) + \varepsilon\boldsymbol{\sigma}^1(\mathbf{x}, y_1) + \varepsilon^2\boldsymbol{\sigma}^2(\mathbf{x}, y_1) + \cdots.$$

Partial differentiation of $\mathbf{u}^k$ and $\boldsymbol{\sigma}^k$ with respect to x_1 need special attention:

$$\partial_1 \rightarrow \partial_1 + \frac{1}{\varepsilon}D_1.$$

We proceed by considering eqns (8), which for this special case is given by

$$(\lambda + 2\mu)D_1 u_1^0 = 0,$$

$$\mu(D_1 u_2^0) + \lambda(D_1 u_1^0) = 0.$$

From these relations, we conclude that $\mathbf{u}^0$ depend only on $\mathbf{x}$ and t.

We also know from eqns (9) that

$$\sigma_{11}^0 = (\lambda + 2\mu)\varepsilon_{11}^0 + \lambda\varepsilon_{22}^0 + (\lambda + \mu)D_1 u_1^1, \tag{23}$$

$$\sigma_{22}^0 = \lambda\varepsilon_{11}^0 + (\lambda + 2\mu)\varepsilon_{22}^0 + \lambda D_1 u_1^1, \tag{24}$$

$$\sigma_{12}^0 = 2\mu\varepsilon_{12}^0 + \lambda D_1 u_1^1, \tag{25}$$

recalling that ε_{ij}^0 is the strain due to u_i^0.

Equation (10) reduces to

$$D_1\sigma_{11}^0 = 0,$$

$$D_1\sigma_{21}^0 = 0,$$

from which we conclude that both σ_{11}^0 and σ_{12}^0 depend only on $\mathbf{x}$ and t. This fact will be useful in the calculation leading to the effective medium. Notice, however, that we cannot say anything about σ_{22}^0. The boundary conditions (20) and (21) imply that

$$c\partial_t(u_2^1|_{a+} - u_2^1|_{a-}) = \sigma_{12}^0(\mathbf{x}, t), \tag{26}$$

$$c\partial_t(u_2^1|_{b+} - u_2^1|_{b-}) = \sigma_{12}^0(\mathbf{x}, t). \tag{27}$$

We shall go through the calculation of the auxiliary functions only for u_1^1. The determination of u_2^1 is much simpler, and does not require the construction of auxiliary functions. From $D_1\sigma_{11}^0 = 0$, we have

$$D_1(\lambda + 2\mu)D_1 u_1^1 = -(\lambda + 2\mu)'\varepsilon_{11}^0 + \lambda'\varepsilon_{22}^0.$$

We have used primes to denote differentiation with respect to y_1. The auxiliary functions χ^{11} and χ^{22} are required to solve

$$D_1(\lambda+2\mu)D_1\chi^{11} = -(\lambda+2\mu)',$$
$$D_1(\lambda+2\mu)D_1\chi^{22} = -\lambda',$$

and be 1-periodic (generalized) functions of y_1. The representation for u_1^1 is

$$D_1 u_1^1 = D_1\chi^{11}\varepsilon_{11}^0 + D_1\chi^{22}\varepsilon_{22}^0.$$

The auxiliary functions χ^{11} and χ^{22} are relatively easy to solve (see Bensoussan *et al.*, 1978, section 2.3). We use the notation

$$\overline{(\cdot)} = \int_{-1/2}^{1/2} \mathrm{d}y_1(\cdot),$$
$$a_1 := \frac{1}{\lambda+2\mu},$$
$$a_2 := \frac{\lambda}{\lambda+2\mu}.$$

The representation for $D_1 u_1^1$ is

$$D_1 u_1^1 = \left[-1+\frac{a_1}{\bar{a}_1}\right]\varepsilon_{11}^0 + \left[-a_2+a_1\frac{\bar{a}_2}{\bar{a}_1}\right]\varepsilon_{22}^0. \tag{28}$$

This result will be inserted into eqns (23) and (25).

To find u_2^1, we take ∂_t of eqn (24) after dividing through by μ. Next, keeping in mind that σ_{12}^0 depends only on x and t, we integrate the expression over the intervals $(-1/2, a-)$, $(a+, b-)$, $(b+, 1/2)$ in y_1 to get

$$\int_{-1/2}^{a-} \mathrm{d}y_1 \frac{1}{\mu}\partial_t\sigma_{12}^0 = (a+\tfrac{1}{2})\partial_t\varepsilon_{12}^0 + \partial_t(u_2^1|_{a-} - u_2^1|_{-1/2}),$$
$$\int_{a+}^{b-} \mathrm{d}y_1 \frac{1}{\mu}\partial_t\sigma_{12}^0 = (b-a)\partial_t\varepsilon_{12}^0 + \partial_t(u_2^1|_{b-} - u_2^1|_{a+}),$$
$$\int_{b+}^{1/2} \mathrm{d}y_1 \frac{1}{\mu}\partial_t\sigma_{12}^0 = (\tfrac{1}{2}-b)\partial_t\varepsilon_{12}^0 + \partial_t(u_2^1|_{1/2} - u_2^1|_{b+}).$$

We add the three expressions, making use of eqns (26) and (27), and the periodicity of u_2^1 to arrive at

$$\overline{\left(\frac{1}{\mu}\right)}\partial_t\sigma_{12}^0 = \partial_t\varepsilon_{12}^0 - \frac{2}{c}\sigma_{12}^0. \tag{29}$$

This is the stress–strain relation governing shear deformation.

Now, we are ready to write down the stress–strain relation for the effective medium. We use eqn (28) in (23) and (25) to get the relation for normal stresses. Hooke's law for the homogenized medium will be written in matrix form:

$$\begin{bmatrix} \sigma^0_{11} \\ \sigma^0_{22} \\ \partial_t\sigma^0_{12} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & 0 \\ 0 & 0 & c_{33} \end{bmatrix} \begin{bmatrix} \varepsilon^0_{11} \\ \varepsilon^0_{22} \\ \partial_t\varepsilon^0_{12} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \gamma\sigma^0_{12} \end{bmatrix}. \tag{30}$$

The matrix elements are

$$c_{11} = \bar{\mu} + \frac{\overline{(\lambda+\mu)a_1}}{\bar{a}_1},$$

$$c_{12} = \overline{\mu a_2} + \frac{\overline{(\lambda+2\mu)a_1\bar{a}_2}}{\bar{a}_1},$$

$$c_{21} = \frac{\bar{a}_2}{\bar{a}_1},$$

$$c_{22} = \overline{4\mu(\lambda+\mu)a_1} + \frac{\bar{a}_2^2}{\bar{a}_1},$$

$$c_{33} = 1\Big/\overline{\left(\frac{1}{\mu}\right)},$$

$$\gamma = -\frac{2}{c}\Big/\overline{\left(\frac{1}{\mu}\right)}.$$

For the momentum balance, we have eqn (20), which we rewrite here for convenience as

$$\bar{\rho}\partial_t^2 u_i^0 = \partial\bar{\sigma}^0_{ij}. \tag{31}$$

Equations (30) and (31) describe the effective medium.

We can now study the property of the stress–strain law. If $c \to \infty$, i.e. perfect bonding of the layers, then the second term on the right-hand-side vanishes. In this case, we get a transversely isotropic medium with five elastic constants.

If the mixture is homogeneous, then $c_{11} = c_{22} = (\lambda+2\mu)$, $c_{12} = c_{21} = c_{33} = \mu$, which corresponds to the elasticities of an isotropic medium. We do get an additional term due to damping. The effect of the damping term will be studied next.

For the homogeneous mixture, we can analyze the solution of the governing partial differential equations using modal analysis. This method will actually work for any anisotropic elastic material, with or without damping, using a modification of the technique used by Synge (1957). However, since the point of this work is to study anisotropic damping, we shall exploit the simplicity afforded by the elastic isotropy.

Let us write down the equations satisfied by the displacement vector $\mathbf{u}(\mathbf{x}, t)$ and the shear stress $\sigma_{12}(\mathbf{x}, t)$. We drop the superscript 0 from now on. From eqns (30) and (31), we have

$$\rho\partial_t^2 u_1 = (\lambda+2\mu)\partial_1^2 u_1 + \lambda\partial_1\partial_2 u_2 + \partial_2\sigma_{12},$$

$$\rho\partial_t^2 u_2 = \lambda\partial_1\partial_2 u_1 + (\lambda+2\mu)\partial_1^2 u_2 + \partial_1\sigma_{12},$$

$$\partial_t\sigma_{12} = \mu\partial_t(\partial_1 u_2 + \partial_2 u_1) - \frac{2\mu}{c}\sigma_{12}.$$

We are interested in a solution of the form

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{v}\, e^{i\mathbf{k}\cdot\mathbf{x} + i\omega(\mathbf{k})t}, \tag{32}$$

a plane wave with a fixed wave number $\mathbf{k} = (k_1, k_2)$, at frequency ω which depends on $\mathbf{k}$.

Substituting eqn (32) into the equations governing $\mathbf{u}$ and σ, we get

$$\begin{bmatrix} r^2k_1^2 + \alpha k_2^2 - \bar{\omega}^2 & (r^2 - 2 + \alpha)k_1k_2 \\ (r^2 - 2 + \alpha)k_1k_2 & r^2k_2^2 + \alpha k_1^2 - \bar{\omega}^2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{33}$$

where we have used the normalized frequency

$$\bar{\omega} = \frac{\omega}{c_s}, \quad c_s = \sqrt{\frac{\mu}{\rho}};$$

and defined the P-to-S wavespeeds ratio (squared)

$$r^2 = \frac{(\lambda + 2\mu)}{\mu},$$

the dissipation constant

$$c' = \frac{2\sqrt{\rho\mu}}{c}$$

and a new variable

$$\alpha = \frac{i\bar{\omega}}{i\bar{\omega} + c'}.$$

The goal now is to find $\bar{\omega}$ and $\mathbf{v}$ which satisfy eqn (33). For each $\mathbf{k}$, these are called the frequency and the displacement mode respectively. The function $\bar{\omega}$ as a function of k is the dispersion curve for that mode. If the dispersion curve has a positive imaginary component, then that mode decays exponentially in time. The imaginary component of $\bar{\omega}$ is referred to as *attenuation*.

For a mode to be non-trivial $\bar{\omega}$ must be such that the determinant in eqn (33) is zero. After writing $\mathbf{k} = (k \cos\theta, k \sin\theta)$, the required condition on the determinant becomes a fifth order polynomial equation:

$$i\bar{\omega}^5 + c'\bar{\omega}^4 - k^2(r^2+1)i\bar{\omega}^2 - k^2c'r^2\bar{\omega}^2 + k^4r^2i\bar{\omega} + k^4c' \sin^2 2\theta(r^2 - 1) = 0. \tag{34}$$

This equation can be solved exactly for $\theta = 0$, i.e. propagation in the x_1 direction. For this angle, we have

$$\begin{aligned} &\bar{\omega} = 0 && \mathbf{v} = (0, 1) \\ &\bar{\omega} = \pm rk && \mathbf{v} = (1, 0) \\ &\bar{\omega} = \pm\sqrt{k^2 - c'^2/4} + ic'/2 && \mathbf{v} = (0, 1). \end{aligned}$$

The first solution corresponds to the static (non-propagating) solution, and we choose to ignore it. The second solution corresponds to P-waves propagating in the x_1 direction. Notice that this mode is *not* attenuated. The last solution corresponds to S-waves, and this mode decays exponentially as a function of time according to a factor of $c'/2$.

The dispersion curves exhibit symmetry about multiples of $\pi/4$ in θ, as exhibited by the last term in eqn (34). For angles other than 0, we resorted to the computer to solve for

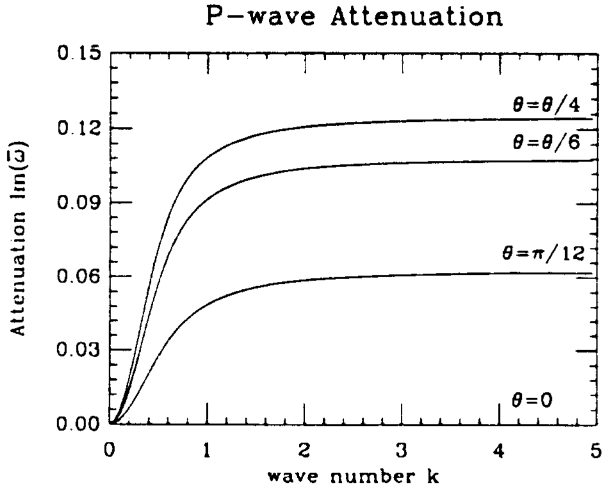

Fig. 3. This figure illustrates the attenuation for waves propagating nearly at the P-wave speed. The constants are $r = 2$, $c' = 1$. Notice that there is no attenuation at $\theta = 0$. Observe also that the attenuation is generally less severe than for S-wave modes.

the roots of eqn (34). For our calculations, we took $r = 2$ and $c' = 1$. We considered wave numbers $0 \leqslant k \leqslant 5$ because the interesting phenomena in the dispersion occur at low wave numbers.

The results show that there is always a static mode. However, this mode decays exponentially for θ not equal to multiples of $\pi/2$. This simply means that if the initial data for an initial value problem begin with a static equilibrium solution, this solution will decay exponentially in time, depending on its Fourier components. However,since we are concerned mostly with propagation problems, this is not an issue.

In addition to static modes, we have two propagating modes. One mode travels nearly at the P-wave speed, and the other at the S-wave speed. Both modes are slightly dispersive near zero wave numbers. The more interesting aspect is their attenuation.

For modes travelling at nearly P-wave speed, we find that the attenuation is zero for $\theta = 0$, as our earlier calculation shows. When θ increases, the attenuation increases and peaks at $\theta = \pi/4$. The behavior is symmetric about $\theta = \pi/4$. For a fixed angle of propagation, the attenuation is small at low wave numbers, then increases until it flattens to a constant value. This is illustrated in Fig. 3.

For modes travelling at nearly S-wave speed (Fig. 4), we find that the attenuation is largest at $\theta = 0$. At this angle, there is a threshold wave number below which no propagating

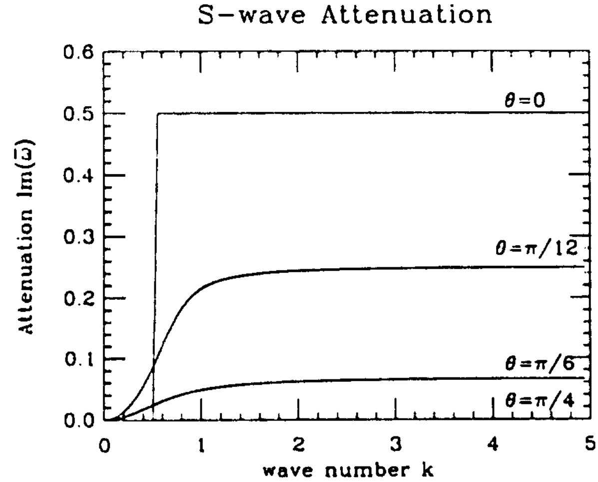

Fig. 4. The attenuation curves for waves propagating nearly at the S-wave speed. The constants are the same as those in Fig. 3. Notice that there is no attenuation at $\theta = \pi/4$.

solution exist. This is displayed in the formula for exact dispersion above. As θ increases, the attenuation decreases until $\theta = \pi/4$, where *no attenuation* occurs. The explanation is that shear waves polarized at $\pi/4$ generate only normal tractions along the interfaces, and since only shear traction produces dissipation, no attenuation occurs.

5. DISCUSSION

We have constructed a model of a medium which is anisotropic in both elasticities and frictional losses. The model material is made up of periodic cells, where each cell is made by mixing two isotropic elastic components. The losses are caused by viscous sliding at the microstructural interfaces.

The technique of homogenization was used to replace the rather complicated structure with an effective medium. It was shown that a new time-dependent stress–strain law is satisfied by the homogenized medium. For the particularly simple case of one-dimensional lamination, we calculated the explicit form of the effective medium equations and studied the properties of the solution.

It should be remarked that we can easily extend the results presented here to the case where the mixture is allowed to vary smoothly from cell to cell. In place of $\mu^{\varepsilon}(\mathbf{x}) = \mu(\mathbf{x}/\varepsilon)$ we would have $\mu^{\varepsilon}(\mathbf{x}) = \mu(\mathbf{x}, \mathbf{x}/\varepsilon)$, and similarly for λ and ρ. The viscous constant may also be made to depend on $\mathbf{x}$. This added complexity only means that the homogenized elasticities and dissipation tensor in eqn (19) will depend on $\mathbf{x}$.

Sensitivity analysis would be a natural test to subject our material to. The analysis will reveal, one hopes, the number and the types of experiments needed to determine all the constants in eqn (19).

As a possible application for the material modeled here, we could consider the problem of finding regions where the dissipation coefficient in equation (19) is substantial from measured responses of the medium to known excitation. This could be a simple-minded caricature of finding zones of debonding (delamination) in a composite medium, as we expect some sliding to occur there.

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